The formation of coherent structures in strongly interacting q-boson systems

1992 J. Phys. A: Math. Gen. 25 2425

(http://iopscience.iop.org/0305-4470/25/9/016)

J. Phys. A: Math. Gen. 25 (1992) 2425-2435. Printed in the UK

The formation of coherent structures in strongly interacting q-boson systems

J A Tuszyński† and M Kibler

Institut de Physique Nucléaire de Lyon IN2P3-CNRS et Université Claude Bernard, 43 Boulevard du 11 Novembre 1918 F-69622 Villeurbanne Cedex, France

Received 20 September 1991

Abstract. We combine two recent developments in the areas of mathematical physics and many body systems. The first is the introduction of systems of q-bosons in connection with the study of quantum groups. The second is the formulation of nonlinear dynamics for interacting many-particle systems. In this paper, we demonstrate the role of quantum groups via the introduction of interacting q-bosons. A number of interesting properties emerge which involve dissipative nonlinear dynamics, bifurcation effects and transitions between localized and extended states.

1. Introduction

In a series of papers (Tuszyński and Dixon 1989a, b, Dixon and Tuszyński 1989, 1990) a new method was developed to treat strongly interacting many-body systems of particles which can be described using the generic effective Hamiltonian

$$H_{\mathrm{eff}}=\sum_{k,l}\omega_{k,l}a_k^\dagger a_l+\sum_{k,l,m}\Delta_{k,l,m}a_k^\dagger a_l^\dagger a_m a_{k+l-m}. \tag{1}$$

Here, the first term describes one-body interactions and the second involves two-body exchanges of energy. The operators $a_k^\dagger$ and a_l are second quantized creators and annihilators, respectively, and they are allowed to be both of Bose-Einstein and Fermi-Dirac type. The latter eventuality, however, is not of interest in the present paper. The labels on the operators k, l and m refer to linear momentum but the method developed does not necessarily rely on this identification. In the present paper we will, instead, identify the operator labels with sites on a real lattice, each site having its own energy structure, and a dispersion relation in both the first and the second terms in equation (1) will provide a coupling between neighbouring lattice sites. The range of applicability of this model is very large and includes (Taylor 1970, Callaway 1976) most phenomena in solid state physics (crystal lattice dynamics, magnetism, electrons in metals, superconductivity, superfluidity, to name but a few examples). It is also quite feasible to extend the range of applications to atomic physics (Judd 1967) as well as to field theory and subatomic phenomena (Fetter and Walecka 1971).

It is the aim of the present paper to extend the validity of the method of coherent structures to the case of quantum groups, i.e. q-deformed quantum oscillators. The approach that will be adopted is to follow the method of coherent structures first, i.e.

† Permanent address: Department of Physics, University of Alberta, Edmonton, Alberta, Canada T6G 2J1.

0305-4470/92/092425+11$04.50

to calculate the Heisenberg equations of motion for the second quantized operators which are now required to obey q-deformed commutation relations. This will be followed by a limitation to include only the nearest-neighbour interactions and then a classical field operator will be introduced to study the dynamics of the envelope. Several important limiting cases will be investigated including the limit $q=1$, q close to 1 (on both sides of unity) and q very large. The resultant equation of motion for the classical field will be derived and solved in a number of important cases.

2. Quantum groups

A new algebraic structure, the structure of quantum group, has been developed since 1985 (Drinfel'd 1985, Jimbo 1985) and is still the subject of extensive developments both in mathematics and theoretical physics. The formal structure of a quantum group is related to the structure of a Hopf bi-algebra and its physical origin is rooted in various fields of theoretical physics (i.e. statistical mechanics, integrable systems, conformal field theory, etc.). In this introductory section of the paper we briefly describe one of the simplest quantum groups, namely the quantum group $\mathrm{SU}_q(2)$.

The starting point is to consider the usual Fock space of quantum states

$$F=\{|n\rangle : n\in N\} \tag{2}$$

which is commonly used in second quantization. Then, linear operators $a^\dagger$, a and N are defined by the following relations:

$$a^\dagger|n\rangle=\sqrt{[n+1]}|n+1\rangle \tag{3a}$$

$$a|n\rangle=\sqrt{[n]}|n-1\rangle \tag{3b}$$

$$N|n\rangle=n|n\rangle \tag{3c}$$

where the symbol $[\ldots]$ is defined through

$$[c]\equiv\frac{q^c-q^{-c}}{q-q^{-1}} \tag{4}$$

for a given (fixed) complex number q and an arbitrary complex number c. The same symbol can be applied to an operator in place of c. Note that in the limiting case $q=1$, we recover the standard definitions of creation, annihilation and number operators for $a^\dagger$, a and N, respectively. We refer to $a^\dagger$, a and N of equation (3) as q-deformed operators.

It can be readily demonstrated that the following properties are satisfied by these operators:

$$(a)^\dagger=a^\dagger \qquad N^+=N \tag{5a}$$

$$[N,a^\dagger]=a^\dagger \qquad [N,a]=-a \tag{5b}$$

where $[X,Y]\equiv[X,Y]_-=XY-YX$ is the commutator of X and Y.

It is also easy to show that

$$aa^\dagger=[N+1] \tag{6a}$$

$$a^\dagger a=[N] \tag{6b}$$

as well as

$$aa^\dagger - q^{-1}a^\dagger a = q^N \tag{7a}$$

$$aa^\dagger - qa^\dagger a = q^{-N}. \tag{7b}$$

Any set $\{a, a^\dagger\}$ of operators which satisfy the properties (6) and (7) is referred to as q-bosons and was first defined by Macfarlane (1989), Biedenharn (1989) and other authors.

The quantum group $SU_q(2)$ can be generated from two commuting sets $\{a, a^\dagger\}$ and $\{b, b^\dagger\}$ of q-bosons. Indeed, it is now well-known that the operators $J_- = b^\dagger a$, $J_3 = (1/2)(N_a - N_b)$ and $J_+ = a^\dagger b$ span the quantum algebra $su_q(2)$ associated with $SU_q(2)$ (Jimbo 1985).

Many other interesting properties can be found for q-bosons which reflect their character as natural extensions of ordinary boson ladder operators. In particular, a q-deformed uncertainty principle is controlled by (Beidenharn 1989)

$$[x, p_x] = \mathrm{i}\hbar([N+1]-[N]) = \mathrm{i}\hbar \frac{\cosh[(n+\frac{1}{2})ln(q)]}{\cosh[\frac{1}{2}ln(q)]}. \tag{8}$$

Furthermore, the q-deformed harmonic oscillator Hamiltonian is defined as (Biedenharn 1989, Macfarlane 1989)

$$H = \tfrac{1}{2}(aa^\dagger + a^\dagger a)\hbar\omega = ([N+1]+[N])\hbar\omega. \tag{9}$$

Similarly, q-deformed angular momentum operators may be obtained and a q-deformation of the Schwinger algebra SO(3, 2) can also be found (Kibler and Négadi 1991).

In this paper we develop another type of extension. We consider an ensemble of *interacting* boson subsystems, labelled by a subscript k, each of which is characterized by the set $\{a_k, a_k^\dagger\}$. We assume that for each k separately relations (6)–(7) are satisfied. In addition, if $k \neq l$ the two sets commute, i.e.

$$[a_k^\dagger, a_l^\dagger] = [a_k, a_l] = [a_k^\dagger, a_l] = 0. \tag{10}$$

We then describe the total system by the Hamiltonian (1) composed of q-bosons.

In the next section we derive the equations of motion for ladder operators using their q-boson properties.

3. Derivation of the equation of motion

We start with the effective Hamiltonian of equation (1) adapted to the problem of q-bosons placed on a lattice where conservation of linear momentum is not involved, i.e. we take

$$H_{\text{eff}} = \sum_{k,l} \omega_{k,l} a_k^\dagger a_l + \sum_{k,l,m,p} \Delta_{k,l,m,p} a_k^\dagger a_l^\dagger a_m a_p \tag{1a}$$

and all the subscripts label lattice sites. Of course, an analogous formulation of the problem for k-space (reciprocal space) purposes is possible but it is more restrictive and hence less general since we then have to impose $p = k + l - m$ as a result of momentum conservation. This will result in some quantum processes (i.e. super exchange) that we discuss later, to be inadmissible. We then calculate the Heisenberg equation of motion for an annihilator a_n of a quantum of energy for the nth q-boson, so that

$$\mathrm{i}\hbar\dot{a}_n = -[H_{\text{eff}}, a_n]. \tag{11}$$

Using the property

$$[AB, C] = [A, C]B + A[B, C] \tag{12}$$

and calculating the commutator $[a, a^\dagger]$ for a q-boson from equation (7) as

$$[a, a^\dagger] = q^N(1+q) + q^{-1}a^\dagger a - qaa^\dagger \tag{13}$$

yield

$$\mathrm{i}\hbar\dot{a}_n = A_n^q\left[\sum_l \omega_{n,l}a_l + \sum_{l,m,p} \Delta_{n,l,m}a_l^\dagger a_m a_p\right] + \sum_{k,m,p} a_k^\dagger A_n^q \Delta_{k,n,m}a_m a_p \tag{14}$$

where we have introduced the operator

$$A_n^q \equiv q^{N_n}(1+q) + q^{-1}a_n^\dagger a_n - qa_n a_n^\dagger. \tag{15}$$

The above operator contains all the crucial information about the q-boson character of the quantum system. It is easy to see that in the limit $q \to 1$, we have $A_n^q \to 1$ and, as a result, equation (14) tends to its standard counterpart, i.e. equation (18) in the paper by Tuszyński and Dixon (1989). Also note that symbols in the two double sums in equation (14) may be conveniently rearranged in order to obtain the simpler formula

$$\mathrm{i}\hbar\dot{a}_n = A_n^q \sum_k \omega_{n,k}a_k + \sum_{k,m,l} (A_n^q\Delta_{n,k,m}a_k^\dagger + a_k^\dagger A_n^q \Delta_{k,n,m})a_m^\dagger a_l. \tag{16}$$

Furthermore, the interchange of $a_k^\dagger$ and $a_l^\dagger$ in the effective Hamiltonian leaves it invariant from which we infer that

$$\Delta_{k,l,m} = \Delta_{l,km} \tag{17}$$

giving the final form

$$\mathrm{i}\hbar\dot{a}_n = A_n^q \sum_k \omega_{n,k}a_k + \sum_{k,m,l} \Delta_{n,k,m}(A_n^q a_k^\dagger + a_k^\dagger A_n^q)a_m^\dagger a_l. \tag{18}$$

In order to study the emergence of coherence in this system and to grasp such phenomena as localization and delocalization, we shall now make several simplifying assumptions about the nature of interactions. First of all, it will be assumed that there is no externally driven flux of energy which leads to the requirement

$$\omega_{n,k} = \omega^0 \qquad \text{if } n = k \tag{19}$$

but there are also hopping terms between nearest neighbours only so that

$$\omega_{n,k} = \omega^1 \qquad \text{if } n = k \pm 1. \tag{20}$$

Next, two-quanta exchanges are also possible between nearest neighbours only and on a given site (local excitations and de-excitations). For simplicity, we shall focus on site n, henceforth referred to as '0' (since the problem has translational invariance and periodic boundary conditions can be imposed) and its nearest neighbour, $n+1$ or $n-1$, referred to as '+' or '−', respectively. In figure 1 we have schematically represented all the one- and two-body terms which are thus retained in the model.

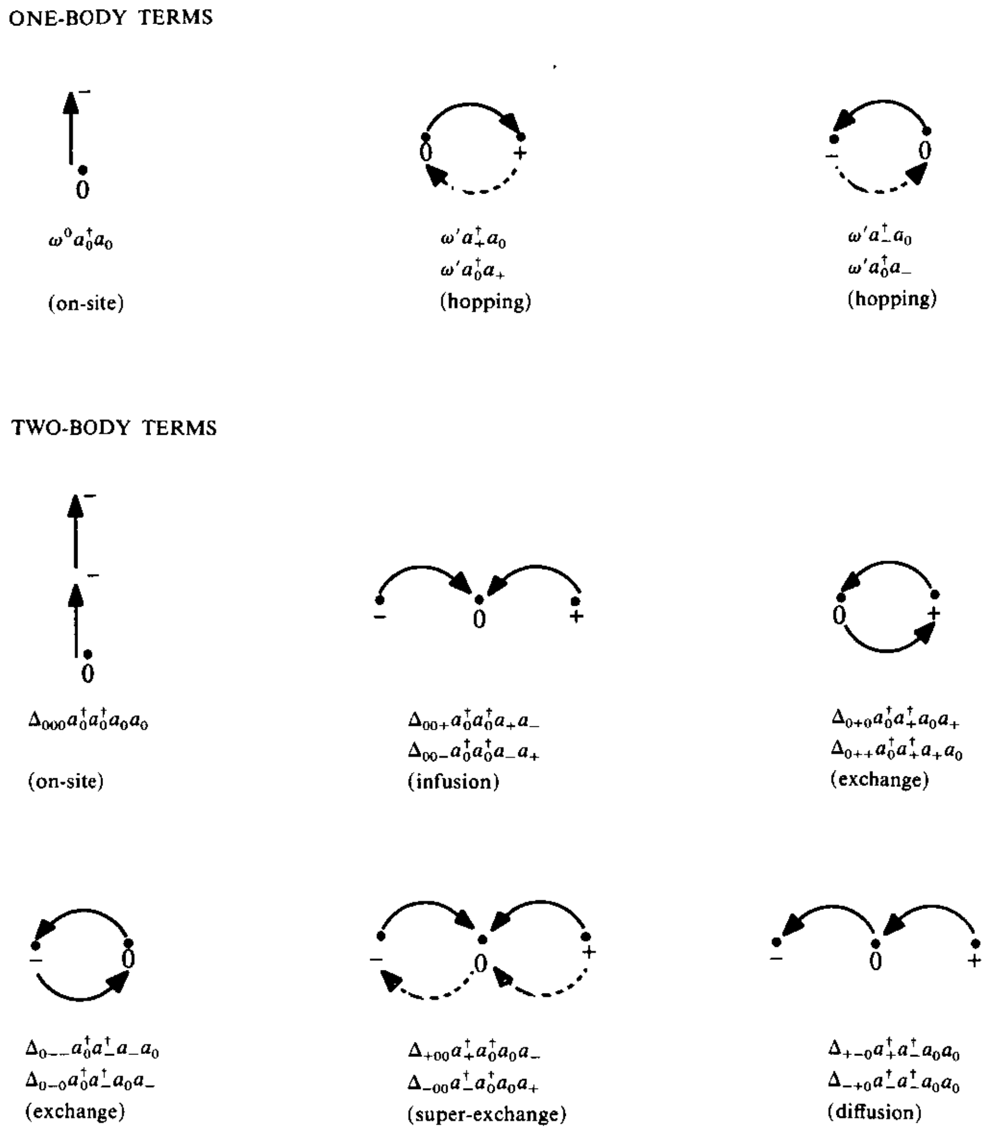

Figure 1. An illustration of the types of quantum processes considered in the model.

4. Field dynamics

Our interest is in obtaining a dynamical description of the classical envelope for the q-boson system in the vicinity of condensation. With this in mind we now invoke the Jackiw-Goldstone formalism (Jackiw 1977, Rajaraman 1987) and represent the quantum field as a linear combination of its classical part (also called field translation) and its quantum component. It is assumed that the latter is a correction on the order of $\hbar$ and, especially in macroscopically occupied states such as is the case with Bose condensation, it can be in the first instance neglected. Subsequently, quantum corrections can be reintroduced by using a linearization procedure with the classical envelope providing an effective binding potential for quantum excitations. Thus, we have:

$$a_0 \simeq \Phi(\boldsymbol{x}) + \Lambda(\boldsymbol{x}) \tag{21}$$

where $\Phi(\boldsymbol{x})$ is a classical field centred at site n and Λ is a quantum field correction. Furthermore, we expect that the neighbouring sites ('+' and '−') which were included in the interactions are separated by a relatively small distance d, so that

$$a_\pm \simeq \Phi(\boldsymbol{x} \pm \boldsymbol{d}) + \Lambda(\boldsymbol{x} \pm \boldsymbol{d}) \tag{22}$$

and the classical field Φ can be approximated through Taylor expansion as

$$\Phi(\boldsymbol{x} \pm \boldsymbol{d}) \simeq \Phi(\boldsymbol{x}) \pm \boldsymbol{d} \cdot \nabla\Phi + \dots . \tag{23}$$

In the next step we derive the equation of motion for the classical envelope Φ making use of the following approximations:

$$a_-^\dagger \simeq \Phi^* - \boldsymbol{d} \cdot \nabla\Phi^* \tag{24a}$$

$$a_- \simeq \Phi - \boldsymbol{d} \cdot \nabla\Phi \tag{24b}$$

$$a_+ \simeq \Phi + \boldsymbol{d} \cdot \nabla\Phi \tag{24c}$$

$$a_+^\dagger \simeq \Phi^* + \boldsymbol{d} \cdot \nabla\Phi^*. \tag{24d}$$

From equation (18), using approximations (21)–(24), keeping only the nearest-neighbour interactions (as described in figure 1) and taking for the number operator

$$N \simeq |\Phi|^2$$

we obtain the following nonlinear differential equation for Φ:

$$\begin{aligned} \mathrm{i}\hbar\Phi_t = {} & [(1+q)\,\mathrm{e}^{\ln q|\Phi|^2} + (q^{-1}-q)|\Phi|^2]\{\omega^0\Phi + \omega^1 d^2\nabla^2\Phi \\ & + 2\Delta_{000}|\Phi|^2\Phi + (\Delta_{00+}+\Delta_{00-})(\Phi^2 - d^2(\nabla\Phi)^2)\Phi^* \\ & + (\Delta_{0+0}+\Delta_{0++})(|\Phi|^2 + d^2|\nabla\Phi|^2)\Phi + (\Delta_{0--}+\Delta_{0-0})(|\Phi|^2 + d^2|\nabla\Phi|^2)\Phi \\ & + (\Delta_{+00}+\Delta_{-00})|\Phi|^2\Phi \end{aligned} \tag{25}$$

where the coefficients Δ are those of equation $(1a)$ with the subscripts 0, + and − being abbreviations for the sites n, $n+1$ and $n-1$, respectively. Various terms can be then grouped for convenience to give

$$\begin{aligned} \mathrm{i}\hbar\Phi_t = {} & [(1+q)\,\mathrm{e}^{\ln q|\Phi|^2} + (q^{-1}-q)|\Phi|^2] \\ & \times\{\Omega_1\Phi + \Omega_2\nabla^2\Phi + \Omega_3|\Phi|^2\Phi + \Omega_4|\nabla\Phi|^2\Phi + \Omega_5(\nabla\Phi)^2\Phi^*\} \end{aligned} \tag{26}$$

where the new coefficients are defined below

$$\Omega_1 = \omega^0$$

$$\Omega_2 = \omega^1 d^2$$

$$\Omega_3 = 2\Delta_{000} + \Delta_{00+} + \Delta_{00-} + \Delta_{0+0} + \Delta_{0++} + \Delta_{0--} + \Delta_{0-0} + \Delta_{-00} + \Delta_{+00}$$

$$\Omega_4 = d^2(\Delta_{0+0} + \Delta_{0++} + \Delta_{0--} + \Delta_{0-0})$$

$$\Omega_5 = -d^2(\Delta_{00+} + \Delta_{00-}).$$

Equation (26), although very complicated, is not impossible to deal with, at least as far as finding special types of solutions. The next section will be devoted to the study of this equation and its analytical solutions.

5. Analysis of the equation of motion

We first note that when $q \to 1$ the term in the square bracket of equation (26) tends to unity also and we obtain a generalized nonlinear Schrödinger equation which has been studied extensively in the past (see for example Gagnon and Winternitz 1988, Dixon and Tuszyński 1989 or Clarkson and Tuszyński 1990). In particular, it has been shown that under special circumstances soliton solutions exist. It will be of special interest to examine the rôle of q in establishing new types of solutions.

The first step in our analysis is to represent the field Φ in modulus-argument form. We thus write

$$\Phi = \eta \, e^{i\psi}. \tag{27}$$

Substituting the above into equation (26) and separating the real and imaginary parts leads to the following two coupled equations

$$\begin{aligned} -\hbar\eta\psi_t = {} & [(1+q)\exp(ln q\eta^2) + (q^{-1}-q)\eta^2]\{\Omega_1\eta + \Omega_2\nabla^2\eta - \Omega_2\eta(\nabla\psi)^2 \\ & + \Omega_3\eta^3 + \Omega_4(\nabla\eta)^3 - \Omega_4\eta^2(\nabla\psi)^2 + \Omega_5\eta(\nabla\eta)^2 - \Omega_5\eta^3(\nabla\psi)^2\} \end{aligned} \tag{28}$$

for the real part, and

$$\hbar\eta_t = [(1+q)\exp(ln q\eta^2) + (q^{-1}-q)\eta^2]\{2\Omega_2\nabla\eta\nabla\psi + \Omega_2\eta\nabla^2\psi + 2\Omega_5\eta^2\nabla\eta\nabla\psi\} \tag{29}$$

for the imaginary part. Note that equation (29) is automatically satisfied when the amplitude η is independent of time and the phase ψ is constant in space, so that $\nabla\psi = \nabla^2\psi = 0$. In this case, which can be referred to as a standing wave (or a breather), equation (28) simplifies to

$$\Omega_2\nabla^2\eta + (\Omega_4 + \Omega_5\eta)(\nabla\eta)^2 = -\Omega_1\eta - \Omega_3\eta^3 - \frac{\hbar\nu\eta}{\{(1+q)\exp[(ln q)\eta^2] + (q^{-1}-q)\eta^2\}} \tag{30}$$

where it was assumed that $\psi = \nu t$.

Several observations can be made based on equation (30). First, for the lowest mode of oscillation, i.e. for $\nu = 0$, the effect of q-bosons on the field dynamics disappears completely since the last term in equation (30) vanishes. The higher the frequency of the temporal phase oscillation, the larger the effect the quantum group has on the field dynamics. In a one-dimensional space equation (30) can be integrated analytically (Vos 1991) since it falls into the class

$$\frac{d^2y}{dx^2} + B(y)\left(\frac{dy}{dx}\right)^2 + C(y) = 0. \tag{31}$$

Through two substitutions:

$$y = \frac{R(z)}{S(z)} \tag{32}$$

and

$$\frac{dz}{dx} = \frac{T(z)}{U(z)} \tag{33}$$

where R, S, T and U are polynomials to be adjusted self-consistently, it can be shown that a solution of equation (31) is given by

$$x - x_0 = \int_{y_0}^{y} dy' \frac{e^{D(y')}}{(-\int_{y_0'}^{y'} dy'' C(y'') \, e^{2D(y'')})^{1/2}} \tag{34}$$

where $D=\int_{y_0}^{y} \mathrm{d}y' B(y')$. In our case, B is a linear function of η and hence D is a quadratic function of η. Thus, the integral under the square root of equation (34) will contain a Gaussian factor and a very complicated combination of polynomial and Gaussian functions. Therefore, indefinite integrals of this type will most likely be impossible to represent in closed analytical form.

In order to gain a qualitative understanding of the rôle played by q, we now analyse the form of the right-hand side of equation (30) (denoted as RHS) in two cases corresponding to $q>0$: (a) $q\approx 1$, on either side of unity, and (b) $q\gg 1$.

(a) If $q\approx 1+\varepsilon$ where $|\varepsilon|\ll 1$ and ε can be positive or negative we can rewrite RHS to the leading order in ε as

$$\mathrm{RHS}\simeq -\left(\Omega_1+\frac{\hbar\nu}{2}\right)\eta-\left(\Omega_3-\frac{\hbar\nu\varepsilon^2}{4}\right)\eta^3. \tag{35}$$

This indicates a rescaling of the model parameters Ω_1 and Ω_3 as a result of q-boson reformulation of the problem for $q\approx 1$. The leading terms appear insensitive to the sign of ε. Since most of the interesting phenomena (bifurcations, separatrix crossing, etc.) occur when the linear term on RHS vanishes, the result will be a shift of this condition proportional to the oscillation frequency ν.

(b) If $q\gg 1$ the situation is quite different since, to the leading order, the right-hand side of equation (30) can be approximated by

$$\mathrm{RHS}\simeq -\Omega_1\eta-\Omega_3\eta^3-\frac{\hbar\nu}{q}\,\eta\,\exp[-(\ln q)\eta^2]. \tag{36}$$

It is easy to see that, with a fixed q, for large amplitudes η we effectively obtain a different type of redressing, namely

$$\mathrm{RHS}\simeq -\left(\Omega_1+\frac{\hbar\nu}{q}\right)\eta-\left(\Omega_3-\frac{\hbar\nu}{q}\ln q\right)\eta^3. \tag{37}$$

It appears that increasing q leads to a diminishing value of 'correction' terms both in the linear and cubic contributions to the right-hand side of equation (37). Since the effects vanish completely as $q\to 1$, we expect an optimum value q_0 of q $(1<q_0<\infty)$, such that the effect on the field dynamics will be the largest. However, on the other side of unity, i.e. for $0<q<1$, a completely different picture emerges. When $q\to 0$, the dominant behaviour can be described by

$$\mathrm{RHS}\simeq -\Omega_1\eta-\Omega_3\eta^3-\frac{\hbar\nu q}{\eta}. \tag{38}$$

Thus, with a fixed non-zero values of q a dramatic effect takes place for small amplitude patterns of η. This can be interpreted as an infinite barrier situated at $\eta=0$ and reflecting all spatial oscillations which would otherwise cross the $\eta=0$ axis.

The importance of investigating RHS can be seen on the following two special cases. When $\Omega_2=\Omega_5=0$ we have the first order equation

$$(\nabla\eta)^2=\frac{1}{\Omega_4}\,\mathrm{RHS}(\eta). \tag{39}$$

We have plotted the various types of behaviour corresponding to this case in figure 2. In figures 2(a), 2(b) we have the situation existing for $q>1$, so that

$$(\nabla\eta)^2 \approx -\Omega_1^{\mathrm{eff}}\eta - \Omega_3^{\mathrm{eff}}\eta^3. \tag{40}$$

with $\Omega_1^{\mathrm{eff}}>0$ and $\Omega_3^{\mathrm{eff}}<0$ in figure 2(a) while $\Omega_1^{\mathrm{eff}}<0$ and $\Omega_3^{\mathrm{eff}}<0$ in figure 2(b). Figures 2(c) and 2(d) refer to the case where $0<q<1$, so that

$$(\nabla\eta)^2 \simeq -\Omega_1^{\mathrm{eff}}\eta - \Omega_3^{\mathrm{eff}}\eta^3 - \Omega_{-1}^{\mathrm{eff}}\eta^{-1}. \tag{41}$$

The dashed regions correspond to non-singular solutions of equation (39). The lowest solution takes the form of a bump soliton (localized in space) while the remaining ones are spatial oscillations (usually in the form of elliptic waves). Note that figures 2(b) and 2(c) correspond to situations where no non-singular solutions are possible.

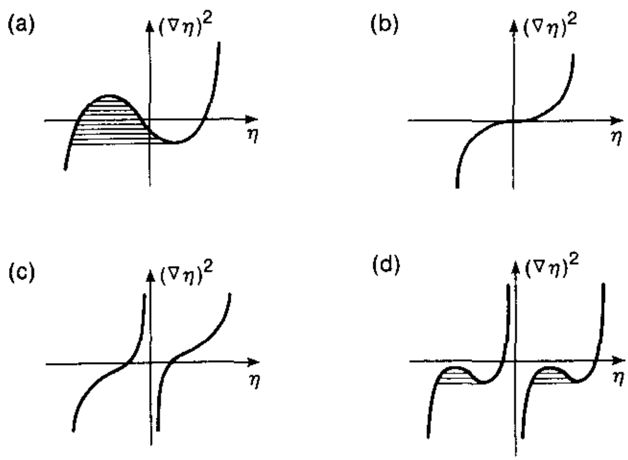

Figure 2. The types of situations encountered in equation (39). (a) $q>1$ and $\Omega_1^{\mathrm{eff}}>0$, $\Omega_3^{\mathrm{eff}}<0$; ($b$) $q<1$ and $\Omega_1^{\mathrm{eff}}<0$, $\Omega_3^{\mathrm{eff}}<0$; ($c$) $0<q<1$ and $\Omega_1^{\mathrm{eff}}<0$, $\Omega_3^{\mathrm{eff}}<0$; ($d$) $0<q<1$ and $\Omega_1^{\mathrm{eff}}>0$, $\Omega_3^{\mathrm{eff}}<0$.

The other special case we wish to discuss occurs when $\Omega_4=\Omega_5=0$, so that we can integrate equation (30) in a one-dimensional space to obtain

$$(\nabla\eta)^2 = \frac{2}{\Omega_2}\int \mathrm{RHS}(\eta)\,\mathrm{d}\eta. \tag{42}$$

This can be in general approximated by:

$$(\nabla\eta)^2 \approx \Omega_2^{\mathrm{eff}}\eta^2 + \Omega_4^{\mathrm{eff}}\eta^4 - \Omega_* \ln|\eta| + \Omega_0 \tag{43}$$

where Ω_0 is an integration constant and $\Omega_*=0$ unless $0<q<1$. Figure 3 illustrates the various types of behaviour. We have assumed that $\Omega_2>0$. An interesting situation occurs in figure 3(e) where an infinitesimal value of q leads to the creation of an infinite number of non-singular wave-type solutions in the case which does not have any non-zero solutions when $q>1$. By analysing these diagrams we clearly see that the behaviour of q-bosons is often, but not always, similar (especially when $q<1$) to that of ordinary interacting bosons.

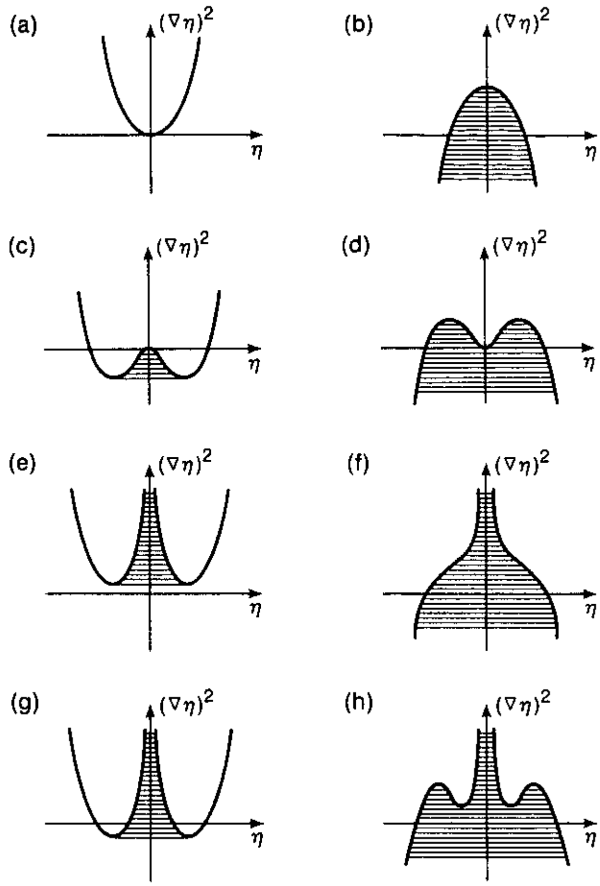

Figure 3. The types of situations encountered in equation (43). Diagrams (a)–(d) refer to $\Omega_* = 0$ while those in (e)–(h) to $\Omega_* = 0$. The latter 4 are arranged in the order to correspond to the former four diagrams. We have (a) $\Omega_2^{\text{eff}} > 0, \Omega_4^{\text{eff}} > 0$; (b) $\Omega_2^{\text{eff}} < 0, \Omega_4^{\text{eff}} < 0$; (c) $\Omega_2^{\text{eff}} < 0$, $\Omega_4^{\text{eff}} > 0$; (d) $\Omega_2^{\text{eff}} > 0$, $\Omega_4^{\text{eff}} < 0$.

6. Conclusions

In the paper we have provided a natural extension of the method of coherent structures from systems of stongly interacting bosons to those composed of interacting q-bosons. In the derivation of field dynamics for the system we have assumed the quantum group extension for their commutation relations and considered mutually commuting sets of q-extended Bose–Einstein operators. Following the derivation of the Heisenberg equation of motion for an annihilation operator, we have retained only local interaction terms and those non-local ones which couple to nearest neighbours only. In spite of this limitation, the obtained field equations for the classical envelope are very rich in structure and non-trivial behaviour. It has been demonstrated that for $q \to 1$ we recover the standard form of the field equations. We then extensively analysed the rôle of the parameter q in obtaining particular types of non-singular solutions. It has been found that for $q \gtrsim 1$ and $q \gg 1$ the corrections merely redress the standard dynamical coefficients and, rather surprisingly, that these corrections diminish as the two extremes are approached. On the other hand, for $0 < q < 1$, a dramatic change is observed when $q \to 0$ with the development of new types of solutions in the previously forbidden range of values. Thus, the q-extension of the method of coherent structures contains some straightforward rescaling (when $q > 1$) as well as highly non-trivial changes when $q < 1$. This suggests the possibility of a bifurcation phenomenon which might occur when q crosses the value 1.

The scope of possible applications of this method is quite large and encompasses anharmonic mode interactions in solids, transport processes along quasi-one-dimensional chains and condensation phenomena in superfluids. We intend to investigate particular applications in the near future.

Acknowledgments

This research was supported by a grant from the Centre Jacques Cartier and also by an NSERC (Canada) operating grant (JAT). One of the authors (JAT) wishes to thank the Staff and Faculty of the Institut de Physique Nucléaire de Lyon for their hospitality during his stay at Villeurbanne.

References

Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
Callaway J 1976 *Quantum Theory of the Solid State* (New York: Academic)
Clarkson P and Tuszyński J A 1990 *J. Phys. A: Math. Gen.* **23** 4269
Dixon J M and Tsuzyński J A 1989 *J. Phys. A: Math. Gen.* **22** 4895
—— 1990 *Physica* B **163** 351
Drinfel'd V G 1985 *Sov. Math. Dokl.* **32** 254
Fetter A L and Walecka J D 1971 *Quantum Theory of Many-Particle Systems* (New York: McGraw-Hill)
Gagnon L and Winternitz P 1988 *J. Phys. A: Math. Gen.* **21** 1493
Jackiw R 1977 *Rev. Mod. Phys.* **49** 681
Jimbo M 1985 *Lett. Math. Phys.* **10** 63
Judd B R 1967 *Second Quantization and Atomic Spectroscopy* (Baltimore, MD: The Johns Hopkins University Press)
Kibler M and Négadi T 1991 *J. Phys. A: Math. Gen.* **24** 5283
Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
Rajaraman R 1987 *Solitons and Instantons. An Introduction to Solitons and Instantons in Quantum Field Theory* (Amsterdam: North-Holland)
Taylor P L 1970 *A Quantum Approach to the Solid State* (New York: Prentice-Hall)
Tuszyński J A and Dixon J M 1989a *J. Phys. A: Math. Gen.* **22** 4877
—— 1989b *Phys. Lett.* **140A** 179
Vos K 1991 private communication